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Liouville equation in the form of a partial differential equation

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Abstract. The Liouville equation for a wide class of boson Hamiltonians can be expressed in the form of partial differential equations for the quasidistributions Φ_N and Φ_A . Their coefficients are calculated by differentiation of the Hamiltonians.

1. Introduction

The time evolution of a quantum system is described by means of the density matrix formalism—the quantum Liouville equation or the Heisenberg equations of motion. Both methods are statistically equivalent. The first one, however, seems more useful for boson systems if Louisell's coherent-state technique (Louisell 1973) is applied. It allows us to study the quantum statistical properties of boson interactions in a c -number domain.

In this paper we show, using the Louisell formalism, that the quantum Liouville equation for a large class of boson interactions can be expressed as partial differential equations for the quasidistribution functions Φ_N and Φ_A . Both quasidistributions are related simply with the density matrix satisfying the Liouville equation. The Φ_N and Φ_A equations we arrive at are formally simple and their coefficients are obtained by differentiation of the Hamiltonian.

2. Equations for the quasidistributions Φ_A and Φ_N

Numerous boson interactions, for example Raman, Brillouin and hyper-Raman scattering, optical parametric processes, and phonon processes in crystals, are governed by the following Hamiltonian (for a review see Peřina 1984)

$$H = H_0 + H_{\text{int}} \quad (2.1)$$

$$H_0 = \hbar \sum \omega_i a_i^\dagger a_i \quad (2.2)$$

$$H_{\text{int}} = \hbar (K Q_1^\dagger Q_2 + K^* Q_2^\dagger Q_1) \quad (2.3)$$

where

$$Q_1^\dagger = a_1^\dagger a_2^\dagger a_3^\dagger \dots a_p^\dagger \quad (2.4)$$

$$Q_2 = a_{p+1} a_{p+2} a_{p+3} \dots a_n.$$

The free Hamiltonian for n oscillators is denoted by H_0 whereas H_{int} plays the role of the effective interaction Hamiltonian. The creation a_i^+ and annihilation a_i operators satisfy the boson commutation relation:

$$[a_i, a_k^+] = \delta_{ik}. \tag{2.5}$$

The parameter K in (2.3) is the coupling constant.

In this paper we show, using Louisell's technique, that the Liouville equation

$$i\hbar \frac{\partial \rho}{\partial t} = [H, \rho] \tag{2.6}$$

for the dynamical system (2.1)-(2.3) can be expressed in the form of partial differential equations:

$$\begin{aligned} i\hbar \frac{\partial \Phi_A}{\partial t} = & \sum_{i=1}^n \left(\frac{\partial}{\partial \alpha_i^*} \left(d_i^* \Phi_A \right) - c c \right) + \sum_{i=1}^n \left(\mathcal{D}_i^* \frac{\partial \Phi_A}{\partial \alpha_i^*} - c c \right) \\ & + \frac{1}{2!} \sum_{i,j=1}^n \left(\mathcal{D}_{ij}^* \frac{\partial^2 \Phi_A}{\partial \alpha_i^* \partial \alpha_j^*} - c c \right) + \frac{1}{3!} \sum_{i,j,k=1}^n \left(\mathcal{D}_{ijk}^* \frac{\partial^3 \Phi_A}{\partial \alpha_i^* \partial \alpha_j^* \partial \alpha_k^*} - c c \right) + \dots \\ & + \frac{1}{r!} \underbrace{\sum_{i,j,k,\dots,l=1}^n}_{r} \left(\mathcal{D}_{ijk\dots l}^* \frac{\partial^r \Phi_A}{\partial \alpha_i^* \partial \alpha_j^* \partial \alpha_k^* \dots \partial \alpha_l^*} - c c \right) \end{aligned} \tag{2.7}$$

or

$$\begin{aligned} i\hbar \frac{\partial \Phi_N}{\partial t} = & \sum_{i=1}^n \left(\frac{\partial}{\partial \alpha_i^*} \left(d_i^* \Phi_N \right) - c c \right) + \sum_{i=1}^n \left(\frac{\partial}{\partial \alpha_i^*} \left(\mathcal{D}_i^* \Phi_N \right) - c c \right) \\ & + \frac{1}{2!} \sum_{i,j=1}^n \left(\frac{\partial^2}{\partial \alpha_i \partial \alpha_j} \left(\mathcal{D}_{ij} \Phi_N \right) - c c \right) + \frac{1}{3!} \sum_{i,j,k=1}^n \left(\frac{\partial^3}{\partial \alpha_i^* \partial \alpha_j^* \partial \alpha_k^*} \left(\mathcal{D}_{ijk}^* \Phi_N \right) - c c \right) \\ & + \dots + \underbrace{\sum_{i,j,k,\dots,l=1}^n}_{r} \left(\frac{(-1)^r}{r!} \frac{\partial^r}{\partial \alpha_i \partial \alpha_j \partial \alpha_k \dots \partial \alpha_l} \left(\mathcal{D}_{ijk\dots l} \Phi_N \right) - c c \right) \end{aligned} \tag{2.8}$$

where the coefficients d and \mathcal{D} are calculated by differentiation of the Hamiltonian (2.1)-(2.3):

$$\begin{aligned} d_i &= \frac{\partial \mathcal{H}_0}{\partial \alpha_i^*} & \mathcal{H}_0 &= \langle \{\alpha_j\} | H_0 | \{\alpha_j\} \rangle \\ \mathcal{D}_i &= \frac{\partial \mathcal{H}_{\text{int}}}{\partial \alpha_i^*} & \mathcal{H}_{\text{int}} &= \langle \{\alpha_j\} | H_{\text{int}} | \{\alpha_j\} \rangle \\ \mathcal{D}_{ij} &= \frac{\partial^2 \mathcal{H}_{\text{int}}}{\partial \alpha_i^* \partial \alpha_j^*} \\ \mathcal{D}_{ijk} &= \frac{\partial^3 \mathcal{H}_{\text{int}}}{\partial \alpha_i^* \partial \alpha_j^* \partial \alpha_k^*} \\ & \vdots \\ \mathcal{D}_{ijk\dots l} &= \frac{\partial^r \mathcal{H}_{\text{int}}}{\partial \alpha_i^* \partial \alpha_j^* \partial \alpha_k^* \dots \partial \alpha_l^*} \end{aligned} \tag{2.9}$$

In the last two terms of the series (2.7) and (2.8) the maximal value of r equals n . The coefficients d^* and \mathcal{D}^* are complex conjugated to d and \mathcal{D} , respectively. The quantity

α_i in (2.7) and (2.8) is defined as an eigenvalue of the annihilation operator a_i , i.e. $a_i|\{\alpha_j\}\rangle = \alpha_i|\{\alpha_j\}\rangle$ where $|\{\alpha_j\}\rangle = \prod_{i=1}^n |\alpha_i\rangle$ is an n -mode coherent state. The quasidistributions Φ_N and Φ_A are used for averaging the normally (N) and antinormally (A) ordered boson operators and are defined as follows (Louisell 1973, Haken *et al* 1967, Lax 1967, Peřina 1984):

$$\Phi_A(\{\alpha_j\}, \{\alpha_j^*\}) = \pi^{-n} \bar{\rho}^N(\{\alpha_j\}, \{\alpha_j^*\}) \tag{2.10}$$

$$\Phi_N(\{\alpha_j\}, \{\alpha_j^*\}) = \pi^{-n} \bar{\rho}^A(\{\alpha_j\}, \{\alpha_j^*\}) \tag{2.11}$$

where n is the number of modes. The function $\bar{\rho}^N(\{\alpha_j\}, \{\alpha_j^*\})$ is defined by

$$\bar{\rho}^N(\{\alpha_j\}, \{\alpha_j^*\}) = \mathcal{N}^{-1} \rho^N(\{a_j\}, \{a_j^+\}) \tag{2.12}$$

where the operator \mathcal{N}^{-1} transforms the density matrix (an operator function) in normal ordering ρ^N to an ordinary function $\bar{\rho}^N$ of the complex variables α, α^* by replacement of a by α and of a^+ by α^* . In order to transform the function $\bar{\rho}^N$ to the operator ρ^N we use the operator \mathcal{N} . To obtain Φ_A in practice we obviously first put ρ into normal form and then write α^* instead of a^+ and α instead of a .

The function $\bar{\rho}^A(\{\alpha_j\}, \{\alpha_j^*\})$ is defined by way of the following relation:

$$\bar{\rho}^A(\{\alpha_j\}, \{\alpha_j^*\}) = \mathcal{A}^{-1} \rho^A(\{a_i\}, \{a_j^+\}) \tag{2.13}$$

where the operator \mathcal{A}^{-1} transforms the antinormally ordered density matrix ρ^A into the classical function $\bar{\rho}^A$. The properties of the operators $\mathcal{N}, \mathcal{N}^{-1}, \mathcal{A}$ and \mathcal{A}^{-1} are considered fully in Louisell (1973, ch 3). Sometimes, in quantum optics, the function Φ_N is referred to as the Glauber-Sudarshan function or P representation and the function Φ_A is denoted by the symbol Q . In general, a Φ_N function does not exist as a tempered distribution for all quantum states, whereas the Φ_A function is always well behaved—it is non-negative and regular. Due to this, the Φ_A function is always preferred in practice (Peřina 1984).

Let us now have a look at the formal procedure of replacing the Liouville equation (2.6) by (2.7). To start with, let us assume that the density matrix ρ in (2.6) can be expressed in normal ordering, i.e. $\rho = \rho^N$. Now, let us take into account the relation (Louisell 1973)

$$[a, f(a, a^+)] = \partial f / \partial a^+ \tag{2.14}$$

leading to

$$af(a, a^+) = f(a, a^+)a + \partial f(a, a^+) / \partial a^+ \tag{2.15}$$

and note that, if $f = f^N$ denotes an expression in normal order, then the right-hand side of (2.15) is normally ordered. This statement allows us to put the commutator

$$[H_0 + H_{int}, \rho^N] \tag{2.16}$$

into normal form. For the first part of the commutator (2.16) we obtain

$$\begin{aligned} [H_0, \rho^N] &= \hbar \sum_{i=1}^n \omega_i (a_i^+ a_i \rho^N - \text{HC}) \\ &= \hbar \sum_{i=1}^n \omega_i \left(a_i^+ \rho^N a_i + \frac{\partial a_i^+ \rho^N}{\partial a_i^+} - \text{HC} \right) \\ &= \hbar \sum_{i=1}^n \omega_i \left(\frac{\partial a_i^+ \rho^N}{\partial a_i^+} - \frac{\partial \rho^N a_i}{\partial a_i} \right). \end{aligned} \tag{2.17}$$

Let us now order the second part of the commutator (2.16):

$$[H_{int}, \rho^N] = \hbar (K Q_1^+ Q_2 \rho^N + K^* Q_2^+ Q_1 \rho^N - \text{HC}) \tag{2.18}$$

to normal form applying the following formula:

$$\begin{aligned}
 Q_1 \rho^N &= Q_1^{(0)} + \sum_{i=1}^p Q_1^{(1)} + \frac{1}{2!} \sum_{i,j=1}^p Q_1^{(2)} + \frac{1}{3!} \sum_{i,j,k=1}^p Q_1^{(3)} + \dots + \frac{1}{r!} \underbrace{\sum_{i,j,k,\dots,l=1}^p}_{r} Q_1^{(r)} \\
 Q_1^{(0)} &= \rho^N Q_1 \\
 Q_1^{(1)} &= \frac{\partial \rho^N}{\partial a_i^+} \frac{\partial Q_1}{\partial a_i} \\
 Q_1^{(2)} &= \frac{\partial^2 \rho^N}{\partial a_i^+ \partial a_j^+} \frac{\partial^2 Q_1}{\partial a_i \partial a_j} \\
 Q_1^{(3)} &= \frac{\partial^3 \rho^N}{\partial a_i^+ \partial a_j^+ \partial a_k^+} \frac{\partial^3 Q_1}{\partial a_i \partial a_j \partial a_k} \\
 &\vdots \\
 Q_1^{(r)} &= \frac{\partial^r \rho^N}{\partial a_i^+ \partial a_j^+ \partial a_k^+ \dots \partial a_l^+} \frac{\partial^r Q_1}{\partial a_i \partial a_j \partial a_k \dots \partial a_l}.
 \end{aligned} \tag{2.19}$$

The proof of (2.19) is given in the appendix.

Taking into account (2.18), we have

$$\begin{aligned}
 \hbar(KQ_1^+ Q_2 + K^* Q_2^+ Q_1) \rho^N - \text{HC} &= \hbar(KQ_1^+ Q_2^{(0)} + K^* Q_2^+ Q_1^{(0)}) - \text{HC} \\
 &+ \hbar \left(K^* \sum_{i=1}^p Q_2^+ Q_1^{(1)} + K \sum_{i=p+1}^n Q_1^+ Q_2^{(1)} \right) - \text{HC} \\
 &+ \frac{\hbar}{2!} \left(K^* \sum_{i,j=1}^p Q_2^+ Q_1^{(2)} + K \sum_{i,j=p+1}^n Q_1^+ Q_2^{(2)} \right) - \text{HC} \\
 &+ \frac{\hbar}{3!} \left(K^* \sum_{i,j,k=1}^p Q_2^+ Q_1^{(3)} + K \sum_{i,j,k=p+1}^n Q_1^+ Q_2^{(3)} \right) - \text{HC} + \dots \\
 &+ \frac{\hbar}{r!} \left(K^* \underbrace{\sum_{i,j,k,\dots,l=1}^p}_{r} Q_2^+ Q_1^{(r)} + K \underbrace{\sum_{i,j,k,\dots,l=p+1}^n}_{r} Q_1^+ Q_2^{(r)} \right) - \text{HC}.
 \end{aligned} \tag{2.20}$$

Let us note that in (2.20) we have two types of sums: in the first $\sum_{i,j,k,\dots,l=1}^p$ the maximal order of the derivatives is $r = p$; in the second $\sum_{i,j,k,\dots,l=p+1}^n$ the maximal order of the derivatives equals $r = n - p$. Since, now, all the terms in the relations (2.17) and (2.20) are normally ordered, we have the Liouville equation (2.6) in the form

$$i \hbar \partial \rho^N / \partial t = (2.17) + (2.20). \tag{2.21}$$

Applying the operator \mathcal{N}^{-1} to both sides of (2.21) and taking into account (2.10) we obtain, after some simple algebra,

$$\begin{aligned}
 i \hbar \frac{\partial \Phi_A}{\partial t} &= \sum_{i=1}^n \frac{\partial}{\partial \alpha_i^*} \left(\frac{\partial \mathcal{H}_0}{\partial \alpha_i} \Phi_A \right) - \text{CC} + \left(\sum_{i=1}^p \frac{\partial \mathcal{H}_{\text{int}}}{\partial \alpha_i} \frac{\partial \Phi_A}{\partial \alpha_i^*} + \sum_{i=p+1}^n \frac{\partial \mathcal{H}_{\text{int}}}{\partial \alpha_i} \frac{\partial \Phi_A}{\partial \alpha_i^*} \right) - \text{HC} \\
 &+ \text{terms higher than first order}
 \end{aligned} \tag{2.22}$$

where

$$\frac{\partial \mathcal{H}_0}{\partial \alpha_i} = \hbar \omega_i \alpha_i^* \quad \mathcal{H}_0 = \langle \{ \alpha_j \} | H_0 | \{ \alpha_j \} \rangle \tag{2.23}$$

$$\sum_{i=1}^p \frac{\partial \mathcal{H}_{\text{int}}}{\partial \alpha_i} \frac{\partial \Phi_A}{\partial \alpha_i^*} = \hbar K^* \sum_{i=1}^p Q_2^* \frac{\partial Q_1}{\partial \alpha_i} \frac{\partial \Phi}{\partial \alpha_i^*} \quad \mathcal{H}_{\text{int}} = \langle \{ \alpha_j \} | H_{\text{int}} | \{ \alpha_j \} \rangle \quad (2.24)$$

$$\sum_{i=p+1}^n \frac{\partial \mathcal{H}_{\text{int}}}{\partial \alpha_i} \frac{\partial \Phi}{\partial \alpha_i^*} = \hbar K \sum_{i=p+1}^n Q_1^* \frac{\partial Q_2}{\partial \alpha_i} \frac{\partial \Phi}{\partial \alpha_i^*}. \quad (2.25)$$

Next, on putting $d_i^* \equiv \partial \mathcal{H}_0 / \partial \alpha_i$ in (2.22), we get the first term of (2.7). The sums $\Sigma_{i=1}^p$ and $\Sigma_{i=p+1}^n$ can be made to collapse, thus allowing us to obtain the term $\Sigma_{i=1}^n \mathcal{D}_i^* \partial \Phi / \partial \alpha_i^*$ in (2.7) with $\mathcal{D}_i^* = \partial \mathcal{H}_{\text{int}} / \partial \alpha_i$. We get the higher terms of (2.7) from (2.22) in the same way.

To find equation (2.8), we assume that the density matrix ρ is expressed in antinormal form and transform the commutator in (2.6) by way of the relation (see appendix)

$$\begin{aligned} Q_1^+ Q_2 \rho^A &= Q_2 \rho^A Q_1^+ - \sum_{i=1}^p \frac{\partial}{\partial a_i} (Q_2 \rho^A) \frac{\partial Q_1^+}{\partial a_i^*} \\ &+ \frac{1}{2!} \sum_{i,j=1}^p \frac{\partial^2 Q_2 \rho^A}{\partial a_i \partial a_j} \frac{\partial^2 Q_1^+}{\partial a_i^* \partial a_j^*} \\ &- \frac{1}{3!} \sum_{i,j,k=1}^p \frac{\partial^3 Q_2 \rho^A}{\partial a_i \partial a_j \partial a_k} \frac{\partial^3 Q_1^+}{\partial a_i^* \partial a_j^* \partial a_k^*} + \\ &+ \frac{(-1)^r}{r!} \underbrace{\sum_{i,j,k,\dots,l=1}^p}_{r} \frac{\partial^r Q_2 \rho^A}{\partial a_i \partial a_j \partial a_k \dots \partial a_l} \frac{\partial^r Q_1^+}{\partial a_i^* \partial a_j^* \partial a_k^* \dots \partial a_l^*}. \end{aligned} \quad (2.26)$$

The other steps are similar to those presented above.

In general, the non-operator equations (2.7) and (2.8) can be said to be two equivalent forms of the Liouville equation (2.6). Although the fact of such equivalence is known (Louisell 1973) and examples of the Φ_N and Φ_A equations have been studied extensively (Peřina 1984), the general form of such equations and their coefficients have as yet not been published. The two equations (2.7) and (2.8) are similar, respectively, to the backward and forward Fokker-Planck equations (Risken 1984). However, our drift-like d_i , \mathcal{D}_i and diffusion-like coefficients $\mathcal{D}_{ijk\dots l}$ do not exhibit the same stochastic properties as in the Fokker-Planck equation itself. Here, they exist independent of all reservoir philosophy connected with the Fokker-Planck equation because their sources are not stochastic Langevin operators but non-linear interactions described by the Hamiltonian (2.1)-(2.3).

3. Conclusion and an example

Let us consider a simple example from quantum optics—the non-degenerate version of parametric lossless generation with classical pumping (Peřina 1984). In this case the Hamiltonian is given by

$$\begin{aligned} H_0 &= \hbar \sum_{i=1}^2 \omega_i a_i^+ a_i \\ H_{\text{int}} &= \hbar g [a_1 a_2 \exp(i\omega t - i\varphi) + \text{HC}]. \end{aligned} \quad (3.1)$$

The corresponding Φ_A equation is given by

$$i\hbar \frac{\partial \Phi_A}{\partial t} = \sum_{i=1}^2 \left(\frac{\partial}{\partial \alpha_i^*} (d_i^* \Phi_A) - c c \right) + \sum_{i=1}^2 \left(\mathcal{D}_i^* \frac{\partial \Phi_A}{\partial \alpha_i^*} - c c \right) + \frac{1}{2!} \sum_{i,j=1}^2 \mathcal{D}_{ij}^* \frac{\partial^2 \Phi_A}{\partial \alpha_i^* \partial \alpha_j^*} - c c \tag{3.2}$$

where the coefficients d and \mathcal{D} are immediately obtained from (2.9)

$$d_i^* = \frac{\partial \mathcal{H}_0}{\partial \alpha_i} = \hbar \omega_i \alpha_i^* \quad i = 1, 2 \quad \mathcal{H}_0 = \sum_{i=1}^2 \hbar \omega_i \alpha_i^* \alpha_i$$

$$\mathcal{D}_1^* = \frac{\partial \mathcal{H}_{int}}{\partial \alpha_1} = \hbar g \alpha_2 \exp(i\omega t - i\varphi) \quad \mathcal{H}_{int} = \hbar g [\alpha_1 \alpha_2 \exp(i\omega t - i\varphi) + c c] \tag{3.3}$$

$$\mathcal{D}_2^* = \frac{\partial \mathcal{H}_{int}}{\partial \alpha_2} = \hbar g \alpha_1 \exp(i\omega t - i\varphi)$$

$$\mathcal{D}_{12}^* = \mathcal{D}_{21}^* = \frac{\partial^2 \mathcal{H}_{int}}{\partial \alpha_1 \partial \alpha_2} = g \hbar \exp(i\omega t - i\varphi).$$

Of course, equations (3.2) and (3.3) can be found with the help of the ‘traditional’ method of Louisell (Peřina 1984), but it is mathematically cumbersome and laborious. The solution of (3.2) and (3.3) has the form

$$\Phi_A(\alpha_1, \alpha_2; t) = [\pi^2 K(t)]^{-1} \exp\left(-\frac{B(t)}{K(t)} \sum_{i=1}^2 |\alpha_i - \xi_i(t)|^2 - \frac{D(t)}{K(t)} [(\alpha_1 - \xi_1(t))(\alpha_2 - \xi_2(t)) + c c] \right) \tag{3.4}$$

where

$$B = \frac{1}{2}(1 - \cosh 2g) + g \cosh 2gt$$

$$\xi_1(t) = \exp(-i\omega_1 t) \xi_1(0) \cosh gt - i \xi_2^*(0) \exp(i\varphi) \sinh gt$$

$$D = -i \exp(-i\omega t) \sinh 2gt$$

$$\xi_2(t) = \exp(-i\omega_2 t) \xi_2(0) \cosh gt - i \xi_1^*(0) \exp(i\varphi) \sinh gt$$

$$K = B^2 - |D|^2$$

which enables us to calculate the statistical moments $\langle a^l a^{+k} \rangle$ as follows (Peřina 1984):

$$\langle a^l a^{+k} \rangle = \int \alpha^l \alpha^{*k} \Phi_A d^2 \alpha. \tag{3.5}$$

It can be proved that the Liouville equation in operator form for the Hamiltonian (3.1) is given by

$$\frac{\partial \rho}{\partial t} = \sum_{j=1}^2 \left(i\omega_j \frac{\partial \rho a_j}{\partial a_j} + \text{HC} \right) + \left[-ig \exp(i\omega t - i\varphi) \left(\frac{\partial \rho a_1}{\partial a_2^+} + \frac{\partial \rho a_2}{\partial a_1^+} + \frac{\partial^2 \rho}{\partial a_1 \partial a_2} \right) + \text{HC} \right] \tag{3.6}$$

and (3.6) does not seem to be easily solvable without the aid of the Φ_A function.

Finally, we see that in practice the method presented of finding the Φ_N and Φ_A equations is 'direct' in that (2.9) leads to the coefficients directly. Our method is applicable to a large class of photon-photon, photon-phonon and other interactions described by Hamiltonians of the type (2.1) and (2.2).

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Appendix

In order to prove (2.19) we start with the following relation (Louisell 1973):

$$[f(a, a^+), a^+] = \partial f(a, a^+) / \partial a \tag{A1}$$

leading to

$$fa^+ = cf \tag{A2}$$

where

$$c = a^+ + \partial / \partial a.$$

We see that if $Q_1 = a_1$ then

$$fQ_1^+ = c_1f = Q_1^+f + \partial f / \partial a_1. \tag{A3}$$

If

$$Q_1 = \prod_{i=1}^2 a_i$$

then

$$fQ_1^+ = \prod_{i=1}^2 c_i f = Q_1^+ f + \sum_{i=1}^2 \frac{\partial Q_1^+}{\partial a_i^+} \frac{\partial f}{\partial a_i} + \frac{1}{2!} \sum_{i,j=1}^2 \frac{\partial^2 f}{\partial a_i \partial a_j}. \tag{A4}$$

If

$$Q_1 = \prod_{i=1}^3 a_i$$

then

$$fQ_1^+ = \prod_{i=1}^3 c_i f = Q_1^+ f + \sum_{i=1}^3 \frac{\partial Q_1^+}{\partial a_i^+} \frac{\partial f}{\partial a_i} + \frac{1}{2!} \sum_{i,j=1}^3 \frac{\partial^2 Q_1^+}{\partial a_i^+ \partial a_j^+} \frac{\partial^2 f}{\partial a_i \partial a_j} + \frac{1}{3!} \sum_{i,j,k=1}^3 \frac{\partial^3 f}{\partial a_i \partial a_j \partial a_k}. \tag{A5}$$

Therefore, if $Q_1 = \prod_{i=1}^p a_i$, the mathematical induction method leads to

$$\begin{aligned}
 Q_1 f Q_1^+ &= \prod_{i=1}^p c_i f = Q_1^+ f + \sum_{i=1}^p \frac{\partial Q_1^+}{\partial a_i^+} \frac{\partial f}{\partial a_i} + \frac{1}{2!} \sum_{i,j=1}^p \frac{\partial^2 Q_1^+}{\partial a_i^+ \partial a_j^+} \frac{\partial^2 f}{\partial a_i \partial a_j} \\
 &+ \frac{1}{3!} \sum_{i,j,k=1}^p \frac{\partial^3 Q_1^+}{\partial a_i^+ \partial a_j^+ \partial a_k^+} \frac{\partial^3 f}{\partial a_i \partial a_j \partial a_k} + \dots \\
 &+ \frac{1}{r!} \underbrace{\sum_{i,j,k,\dots,l=1}^p}_{r} \frac{\partial^r Q_1^+}{\partial a_i^+ \partial a_j^+ \partial a_k^+ \dots \partial a_l^+} \frac{\partial^r f}{\partial a_i \partial a_j \partial a_k \dots \partial a_l}.
 \end{aligned}
 \tag{A6}$$

In order to get (2.19) from (A6), we have to replace the operator function f by normally ordered matrix ρ^N and then conjugate both sides of (A6).

The relation (2.24) can be derived similarly as (A6) if we start with

$$fa = cf \tag{A7}$$

were $c = a - \partial/\partial a^+$.

Note added in proof. Recently, relevant results concerning the quantum Liouville equation for the Φ_A function of the anharmonic oscillator have been proposed by Milburn (1986).

References

Haken H, Risken H and Weidlich W 1967 *Z. Phys.* **206** 355
 Lax M 1967 *Dynamical Processes in Solid State Optics* ed R Kubo and H Kamimura (New York: Benjamin) p 195
 Louisell W H 1973 *Quantum Statistical Properties of Radiation* (New York: Wiley)
 Milburn G J 1986 *Phys. Rev. A* **33** 674
 Peřina J 1984 *Quantum Statistics of Linear and Nonlinear Optical Phenomena* (Dordrecht: Reidel)
 Risken H 1984 *The Fokker-Planck Equation* (Berlin: Springer)